

Solution

INTRODUCTION TO TRIGONOMETRY WS 8

Class 10 - Mathematics

Section A

1. 1

Explanation:

Given:

$$\begin{aligned} \operatorname{cosec}^2 \theta (1 + \cos \theta)(1 - \cos \theta) &= \lambda \\ \Rightarrow \operatorname{cosec}^2 \theta \{(1 + \cos \theta)(1 - \cos \theta)\} &= \lambda \\ \Rightarrow \operatorname{cosec}^2 \theta (1 - \cos^2 \theta) &= \lambda \\ \Rightarrow \operatorname{cosec}^2 \theta \sin^2 \theta &= \lambda \\ \Rightarrow \frac{1}{\sin^2 \theta} \times \sin^2 \theta &= \lambda \\ \Rightarrow 1 &= \lambda \\ \Rightarrow \lambda &= 1 \end{aligned}$$

Thus, the value of λ is 1.

2. 60

Explanation:

$$\text{Given, } \frac{\cos \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 + \sin \theta} = 4$$

Taking LCM

$$\begin{aligned} \frac{\cos \theta (1 + \sin \theta) + \cos \theta (1 - \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} &= 4 \\ \frac{\cos \theta [1 + \sin \theta + 1 - \sin \theta]}{1 - \sin^2 \theta} &= 4 \\ \frac{\cos \theta (2)}{\cos^2 \theta} &= 4 \\ \frac{2}{\cos \theta} &= 4 \\ \cos \theta &= \frac{2}{4} = \frac{1}{2} \\ \cos \theta &= \cos 60^\circ \\ \therefore \theta &= 60^\circ \end{aligned}$$

3. 2

Explanation:

We have,

$$\begin{aligned} \tan \theta + \cot \theta &= 2 \\ \Rightarrow (\tan \theta + \cot \theta)^2 &= 4 \text{ [On squaring both sides]} \\ \Rightarrow \tan^2 \theta + \cot^2 \theta + 2 \tan \theta \cot \theta &= 4 \\ \Rightarrow \tan^2 \theta + \cot^2 \theta + 2 &= 4 \text{ [}\because \tan \theta \cot \theta = 1\text{]} \\ \Rightarrow \tan^2 \theta + \cot^2 \theta &= 2 \end{aligned}$$

4. 90

Explanation:

$$\begin{aligned} \text{Given: } 2 \sin^2 \theta - \cos^2 \theta &= 2 \\ \Rightarrow 2(1 - \cos^2 \theta) - \cos^2 \theta &= 2 \text{ [}\because \sin^2 \theta + \cos^2 \theta = 1\text{]} \\ \Rightarrow 2 - 2 \cos^2 \theta - \cos^2 \theta &= 2 \\ \Rightarrow 2 - 3 \cos^2 \theta &= 2 \\ \Rightarrow -3 \cos^2 \theta &= 2 - 2 \\ \Rightarrow -3 \cos^2 \theta &= 0 \\ \Rightarrow \cos^2 \theta &= 0 \\ \Rightarrow \cos \theta &= 0 \\ \Rightarrow \cos \theta &= \cos 90^\circ \\ \Rightarrow \theta &= 90^\circ \end{aligned}$$

5. (a) - (ii), (b) - (i), (c) - (iv), (d) - (iii)

6. (a) - (iii), (b) - (i), (c) - (iv), (d) - (ii)

7. (a) - (iii), (b) - (i), (c) - (ii), (d) - (iv)

Section B

8. Given, $\tan \theta + \sec \theta = l \dots\dots(1)$

We know that, $\sec^2 \theta - \tan^2 \theta = 1 \dots\dots(2)$

Now, $\sec \theta + \tan \theta = l$ [from (1)]

$\Rightarrow (\sec \theta + \tan \theta) \frac{(\sec \theta - \tan \theta)}{\sec \theta - \tan \theta} = 1$

$\Rightarrow \frac{\sec^2 \theta - \tan^2 \theta}{\sec \theta - \tan \theta} = l$

$\Rightarrow \frac{1}{\sec \theta - \tan \theta} = l$ [from equation (2)]

or, $\sec \theta - \tan \theta = \frac{1}{l} \dots\dots(3)$

Now, to get $\sec \theta$, eliminating $\tan \theta$ from (1) and (3)

adding (1) and (3) we get :-

$\Rightarrow 2 \sec \theta = l + \frac{1}{l}$

$\Rightarrow 2 \sec \theta = \frac{l^2 + 1}{l}$

$\Rightarrow \sec \theta = \frac{l^2 + 1}{2l}$

Hence, proved.

9. We have,

$\tan \theta + \frac{1}{\tan \theta} = 2$

Squaring both sides, we get

$\Rightarrow \left(\tan \theta + \frac{1}{\tan \theta} \right)^2 = 2^2$

$\Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} + 2 \times \tan \theta \times \frac{1}{\tan \theta} = 4$

$\Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} + 2 = 4$

$\Rightarrow \tan^2 \theta + \frac{1}{\tan^2 \theta} = 2$

Alternate method, We have

$\tan \theta + \frac{1}{\tan \theta} = 2$

$\Rightarrow \tan^2 \theta + 1 = 2 \tan \theta$

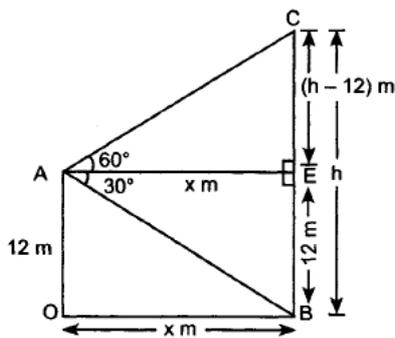
$\Rightarrow \tan^2 \theta - 2 \tan \theta + 1 = 0$

$\Rightarrow (\tan \theta - 1)^2 = 0$

$\Rightarrow \tan \theta = 1$

$\therefore \tan^2 \theta + \frac{1}{\tan^2 \theta} = 1 + 1 = 2$

10. A is the position of the man, OA = 12 m, BC is cliff.



BC = h m and CE = (h - 12) m

Let AE = OB = x m

In right angled triangle AEB,

$\frac{AE}{BE} = \cot 30^\circ \Rightarrow AE = 12 \times \sqrt{3}$
 $= 12 \times 1.732 \text{ m} = 20.78 \text{ m}$

\therefore Distance of ship from cliff = 20.78 m.

In right angled triangle AEC,

$\frac{CE}{AE} = \tan 60^\circ \Rightarrow \frac{h-12}{12\sqrt{3}} = \sqrt{3}$

$h - 12 = 36 \Rightarrow h = 48 \text{ m}$

11. LHS

$$\begin{aligned}
&= (\sin A - \sec A)^2 + (\cos A - \operatorname{cosec} A)^2 = \left(\sin A - \frac{1}{\cos A}\right)^2 + \left(\cos A - \frac{1}{\sin A}\right)^2 \\
&= \left(\frac{\sin A \cos A - 1}{\cos A}\right)^2 + \left(\frac{\sin A \cos A - 1}{\sin A}\right)^2 \\
&= \frac{(\sin A \cos A - 1)^2}{\cos^2 A} + \frac{(\sin A \cos A - 1)^2}{\sin^2 A} \\
&= \frac{\sin^2 A (\sin A \cos A - 1)^2 + \cos^2 A (\sin A \cos A - 1)^2}{\sin^2 A \cos^2 A} \\
&= \frac{(\sin A \cos A - 1)^2 (\sin^2 A + \cos^2 A)}{\sin^2 A \cos^2 A} \\
&= \frac{\sin^2 A \cos^2 A + 1 - 2 \sin A \cos A}{\sin^2 A \cos^2 A} \quad [\text{Since, } \sin^2 A + \cos^2 A = 1] \\
&= \frac{\sin^2 A \cos^2 A}{\sin^2 A \cos^2 A} + \frac{1}{\sin^2 A \cos^2 A} - \frac{2 \sin A \cos A}{\sin^2 A \cos^2 A} \\
&= 1 + \sec^2 A \cdot \operatorname{cosec}^2 A - 2 \sec A \cdot \operatorname{cosec} A \\
&= (1 - \sec A \cdot \operatorname{cosec} A)^2 \\
&= \text{RHS}
\end{aligned}$$

12. We have $\frac{\cos^2(45^\circ + \theta) + \cos^2(45^\circ - \theta)}{\tan(60^\circ + \theta) \tan(30^\circ - \theta)} + \operatorname{cosec}(75^\circ + \theta) - \sec(15^\circ - \theta)$

$$\begin{aligned}
&= \frac{\cos^2(45^\circ + \theta) + \sin^2(90^\circ - 45^\circ + \theta)}{\tan(60^\circ + \theta) \cot(90^\circ - 30^\circ + \theta)} + \operatorname{cosec}(75^\circ + \theta) - \operatorname{cosec}(90^\circ - 15^\circ + \theta) \quad [\because \operatorname{cosec}(90^\circ - \theta) = \sec \theta, \sin(90^\circ - \theta) = \cos \theta] \\
&= \frac{\cos^2(45^\circ + \theta) + \sin^2(45^\circ + \theta)}{\tan(60^\circ + \theta) \cot(60^\circ + \theta)} + \operatorname{cosec}(75^\circ + \theta) - \operatorname{cosec}(75^\circ + \theta) \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

13. We have, $p = \sin \theta + \cos \theta$ and $q = \sec \theta + \operatorname{cosec} \theta$

$$\begin{aligned}
\therefore \text{LHS} &= q(p^2 - 1) = (\sec \theta + \operatorname{cosec} \theta) \{(\sin \theta + \cos \theta)^2 - 1\} \\
&= \left(\frac{1}{\cos \theta} + \frac{1}{\sin \theta}\right) \{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta - 1\} \\
&= \left(\frac{\sin \theta + \cos \theta}{\cos \theta \sin \theta}\right) (1 + 2 \sin \theta \cos \theta - 1) \\
&= \left(\frac{\sin \theta + \cos \theta}{\cos \theta \sin \theta}\right) (2 \sin \theta \cos \theta) = 2(\sin \theta + \cos \theta) = 2p = \text{RHS}
\end{aligned}$$

14. Given,

$$\begin{aligned}
&\cot \theta + \tan \theta = x \text{ and } \sec \theta - \cos \theta = y \\
\Rightarrow &\frac{1}{\tan \theta} + \tan \theta = x \text{ and } \frac{1}{\cos \theta} - \cos \theta = y \\
\Rightarrow &\frac{1 + \tan^2 \theta}{\tan \theta} = x \text{ and } \frac{1 - \cos^2 \theta}{\cos \theta} = y \\
\Rightarrow &\frac{\sec^2 \theta}{\tan \theta} = x \text{ and } \frac{\sin^2 \theta}{\cos \theta} = y \\
\Rightarrow &\frac{1}{\cos^2 \theta \times \frac{\sin \theta}{\cos \theta}} = x \text{ and } \frac{\sin^2 \theta}{\cos \theta} = y \\
\Rightarrow &\frac{1}{\cos \theta \sin \theta} = x \text{ and } \frac{\sin^2 \theta}{\cos \theta} = y \dots\dots\dots \text{(A)}
\end{aligned}$$

Now, L.H.S.

$$\begin{aligned}
&= (x^2 y)^{2/3} - (x y^2)^{2/3} \\
&= \left\{ \frac{1}{\cos^2 \theta \sin^2 \theta} \times \frac{\sin^2 \theta}{\cos \theta} \right\}^{2/3} - \left\{ \frac{1}{\cos \theta \sin \theta} \times \frac{\sin^4 \theta}{\cos^2 \theta} \right\}^{2/3} \quad [\text{From (A)}] \\
&= \left(\frac{1}{\cos^3 \theta}\right)^{2/3} - \left(\frac{\sin^3 \theta}{\cos^3 \theta}\right)^{2/3} \\
&= \frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} = \sec^2 \theta - \tan^2 \theta \\
&= 1 = \text{R.H.S.}
\end{aligned}$$

Hence, Proved.

15. LHS

$$\begin{aligned}
&= \sin^6 \theta + \cos^6 \theta \\
&= (\sin^2 \theta)^3 + (\cos^2 \theta)^3 \\
&= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) \\
&\quad [\because a^3 + b^3 = (a + b)^3 - 3ab(a + b)] \\
&= (1)^3 - 3 \sin^2 \theta \cos^2 \theta \times 1 \quad [\because \sin^2 \theta + \cos^2 \theta = 1] \\
&= 1 - 3 \sin^2 \theta \cos^2 \theta \\
\text{RHS} &= \frac{4 - 3(x^2 - 1)^2}{4}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4-3\{(\sin\theta+\cos\theta)^2-1\}^2}{4} \quad [\text{given } x = \sin\theta + \cos\theta] \\
&= \frac{4-3\{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1\}^2}{4} \\
&= \frac{4-3\{1+2\sin\theta\cos\theta-1\}^2}{4} \quad [\because \sin^2\theta + \cos^2\theta = 1] \\
&= \frac{4-3(2\sin\theta\cos\theta)^2}{4} \\
&= \frac{4-3 \times 4\sin^2\theta\cos^2\theta}{4} \\
&= \frac{4(1-3\sin^2\theta\cos^2\theta)}{4}
\end{aligned}$$

LHS = RHS

Hence proved.

16. Given $3\sin\theta + 5\cos\theta = 5$

$$\begin{aligned}
&\Rightarrow 3\sin\theta = 5 - 5\cos\theta \\
&\Rightarrow 3\sin\theta = 5(1 - \cos\theta) \\
&\Rightarrow 3\sin\theta = \frac{5(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)} \\
&\Rightarrow 3\sin\theta = \frac{5(1-\cos^2\theta)}{1+\cos\theta} \\
&\Rightarrow 3\sin\theta = \frac{5\sin^2\theta}{1+\cos\theta} \quad [\because 1 - \cos^2\theta = \sin^2\theta] \\
&\Rightarrow 3 = \frac{5\sin\theta}{1+\cos\theta} \\
&\Rightarrow 3 + 3\cos\theta = 5\sin\theta \\
&\Rightarrow 3 = 5\sin\theta - 3\cos\theta
\end{aligned}$$

Hence proved.

17. We have to prove :-

$$\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta.$$

Now, take LHS = $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}$

$$\begin{aligned}
&= \sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}} = \sqrt{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta}} \\
&= \sqrt{\frac{1}{\sin^2\theta \cdot \cos^2\theta}} = \frac{1}{\sin\theta \cos\theta} \\
&= \operatorname{cosec}\theta \sec\theta \dots (1)
\end{aligned}$$

Now, take RHS = $\tan\theta + \cot\theta$

$$\begin{aligned}
&= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cos\theta} \\
&= \frac{1}{\sin\theta \cdot \cos\theta} \\
&= \operatorname{cosec}\theta \cdot \sec\theta \dots (2)
\end{aligned}$$

Hence, from (1) & (2)

LHS=RHS, Proved.

18. To prove-

$$\frac{1}{(\cot A)(\sec A) - \cot A} - \operatorname{cosec} A = \operatorname{cosec} A - \frac{1}{(\cot A)(\sec A) + \cot A}$$

Taking LHS

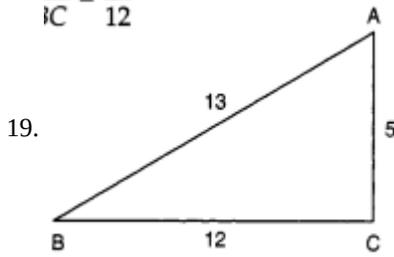
$$\begin{aligned}
&= \frac{1}{(\cot A)(\sec A) - \cot A} - \operatorname{cosec} A \\
&= \frac{1}{\left(\frac{\cos A}{\sin A}\right)\left(\frac{1}{\cos A}\right) - \left(\frac{\cos A}{\sin A}\right)} - \frac{1}{\sin A} \\
&= \frac{1}{\left(\frac{1}{\sin A}\right) - \left(\frac{\cos A}{\sin A}\right)} - \frac{1}{\sin A} = \frac{1}{\frac{1-\cos A}{\sin A}} - \frac{1}{\sin A} = \frac{\sin A}{1-\cos A} - \frac{1}{\sin A} = \frac{\sin^2 A - 1 + \cos A}{(1-\cos A)\sin A} \\
&= \frac{-\cos^2 A + \cos A}{(1-\cos A)\sin A} = \frac{\cos A(1-\cos A)}{(1-\cos A)\sin A} \quad \{\because \sin^2 A + \cos^2 A = 1\} \\
&= \frac{\cos A}{\sin A} = \cot A
\end{aligned}$$

Now, taking RHS

$$\begin{aligned}
&= \operatorname{cosec} A - \frac{1}{(\cot A)(\sec A) + \cot A} \\
&= \frac{1}{\sin A} - \frac{1}{\left(\frac{\cos A}{\sin A}\right)\left(\frac{1}{\cos A}\right) + \frac{\cos A}{\sin A}} \\
&= \frac{1}{\sin A} - \frac{1}{\left(\frac{1}{\sin A}\right) + \frac{\cos A}{\sin A}} = \frac{1}{\sin A} - \frac{\sin A}{(1+\cos A)} \\
&= \frac{1+\cos A - \sin^2 A}{(1+\cos A)\sin A} = \frac{\cos^2 A + \cos A}{(1+\cos A)\sin A}
\end{aligned}$$

$$= \frac{\cos A(\cos A + 1)}{(1 + \cos A) \sin A} = \frac{\cos A}{\sin A}$$

$$= \cot A = \text{LHS}$$



We have

$$\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{12}{5}$$

Let us draw a right triangle ABC, in which $\angle C = 90^\circ$ such that Base = BC = 12 units and, Perpendicular = AC = 5 units.

Applying Pythagoras Theorem in $\triangle BCA$ we get

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow AB^2 = 12^2 + 5^2 = 169$$

$$\Rightarrow AB = \sqrt{169} = 13$$

$$\therefore \sin B = \frac{AC}{AB} = \frac{5}{13}, \tan B = \frac{AC}{BC} = \frac{5}{12} \text{ and, } \sec B = \frac{AB}{BC} = \frac{13}{12}$$

Now, L.H.S = $\tan^2 B - \sin^2 B$

$$\Rightarrow \text{L. H. S} = (\tan B)^2 - (\sin B)^2$$

$$\Rightarrow \text{L. H. S} = \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{25}{144} - \frac{25}{169}$$

$$\Rightarrow \text{L. H. S} = 25 \left(\frac{1}{144} - \frac{1}{169}\right) = 25 \left(\frac{169-144}{144 \times 169}\right)$$

$$\Rightarrow \text{L. H. S} = 25 \times \frac{25}{144 \times 169} = \frac{25 \times 25}{144 \times 169} = \frac{5^2 \times 5^2}{12^2 \times 13^2} \dots\dots(i)$$

and, R. H. S = $\sin^4 B \cdot \sec^2 B$

$$\Rightarrow \text{R. H. S} = (\sin B)^4 (\sec B)^2 = \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2 = \frac{5^4 \times 13^2}{13^4 \times 12^2} = \frac{5^4}{13^2 \times 12^2} = \frac{5^2 \times 5^2}{13^2 \times 12^2} \dots\dots(ii)$$

From (i) and (ii), we have

$$\tan^2 B - \sin^2 B = \sin^4 B \cdot \sec^2 B$$

Hence proved.

20. Taking L.H.S. : $\frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta}$

$$= \frac{\sin \theta (1 - 2 \sin^2 \theta)}{\cos \theta (2 \cos^2 \theta - 1)}$$

$$= \frac{\sin \theta (\sin^2 \theta + \cos^2 \theta - 2 \sin^2 \theta)}{\cos \theta (2 \cos^2 \theta - \sin^2 \theta - \cos^2 \theta)} [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$= \frac{\tan \theta (\cos^2 \theta - \sin^2 \theta)}{(\cos^2 \theta - \sin^2 \theta)}$$

$$= \tan \theta$$

Hence Proved, LHS = RHS

21. Given,

$$\tan A = n \tan B$$

$$\Rightarrow \tan B = \frac{1}{n} \tan A$$

$$\Rightarrow \cot B = \frac{n}{\tan A} \dots\dots(1)$$

Also given,

$$\sin A = m \sin B$$

$$\Rightarrow \sin B = \frac{1}{m} \sin A$$

$$\Rightarrow \operatorname{cosec} B = \frac{m}{\sin A} \dots\dots(2)$$

We know that, $\operatorname{cosec}^2 B - \cot^2 B = 1$, hence from (1) & (2) :-

$$\frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$

$$\Rightarrow \frac{m^2}{\sin^2 A} - \frac{n^2 \cos^2 A}{\sin^2 A} = 1$$

$$\Rightarrow \frac{m^2 - n^2 \cos^2 A}{\sin^2 A} = 1$$

$$\begin{aligned} \Rightarrow m^2 - n^2 \cos^2 A &= \sin^2 A \\ \Rightarrow m^2 - n^2 \cos^2 A &= 1 - \cos^2 A \\ \Rightarrow m^2 - 1 &= n^2 \cos^2 A - \cos^2 A \\ \Rightarrow m^2 - 1 &= (n^2 - 1) \cos^2 A \\ \Rightarrow \frac{m^2 - 1}{n^2 - 1} &= \cos^2 A \end{aligned}$$

22. By the given condition of question

$$\sec \theta = x + \frac{1}{4x}$$

$$\therefore \tan^2 \theta = \sec^2 \theta - 1$$

$$\Rightarrow \tan^2 \theta = \left(x + \frac{1}{4x}\right)^2 - 1 = x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1 = x^2 + \frac{1}{16x^2} - \frac{1}{2} = \left(x - \frac{1}{4x}\right)^2$$

$$\Rightarrow \tan \theta = \pm \left(x - \frac{1}{4x}\right)$$

$$\Rightarrow \tan \theta = \left(x - \frac{1}{4x}\right) \text{ or, } \tan \theta = -\left(x - \frac{1}{4x}\right)$$

CASE 1: When $\tan \theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec \theta + \tan \theta = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$$

CASE 2: When $\theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec \theta + \tan \theta = \left(x + \frac{1}{4x}\right) - \left(x - \frac{1}{4x}\right) = \frac{2}{4x} = \frac{1}{2x}$$

Hence, $\sec \theta + \tan \theta = 2x$ or, $\frac{1}{2x}$

23. We have, $4 \tan \theta = 3$

$$\Rightarrow \tan \theta = \frac{3}{4}$$

$$\therefore \sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

Now, $\frac{4 \sin \theta - \cos \theta + 1}{4 \sin \theta + \cos \theta - 1}$

$$\begin{aligned} &= \frac{\cos \theta \left(4 \frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\cos \theta} + \frac{1}{\cos \theta}\right)}{\cos \theta \left(4 \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta} - \frac{1}{\cos \theta}\right)} \\ &= \frac{4 \tan \theta - 1 + \sec \theta}{4 \tan \theta + 1 - \sec \theta} \end{aligned}$$

Substituting the values, we get,

$$\begin{aligned} &= \frac{4\left(\frac{3}{4}\right) - 1 + \left(\frac{5}{4}\right)}{4\left(\frac{3}{4}\right) + 1 - \left(\frac{5}{4}\right)} = \frac{3 - 1 + \left(\frac{5}{4}\right)}{3 + 1 - \left(\frac{5}{4}\right)} = \frac{2 + \left(\frac{5}{4}\right)}{4 - \left(\frac{5}{4}\right)} = \frac{\frac{8+5}{4}}{\frac{16-5}{4}} = \frac{\frac{13}{4}}{\frac{11}{4}} = \frac{13}{4} \times \frac{4}{11} = \frac{13}{11} \end{aligned}$$

24. LHS

$$\begin{aligned} &= \frac{\sin \theta}{1 - \cos \theta} + \frac{\tan \theta}{1 + \cos \theta} \\ &= \frac{\sin \theta}{1 - \cos \theta} + \frac{\frac{\sin \theta}{\cos \theta}}{1 + \cos \theta} \\ &= \frac{\sin \theta}{1 - \cos \theta} + \frac{\sin \theta}{\cos \theta (1 + \cos \theta)} \\ &= \frac{\sin \theta \cdot \cos \theta (1 + \cos \theta) + \sin \theta (1 - \cos \theta)}{(1 - \cos \theta) \cos \theta (1 + \cos \theta)} \quad [\text{taking LCM}] \\ &= \frac{\sin \theta \cdot \cos \theta + \sin \theta \cdot \cos^2 \theta + \sin \theta - \sin \theta \cdot \cos \theta}{\cos \theta (1 - \cos^2 \theta)} \quad [\text{Since, } (a-b)(a+b) = a^2 - b^2] \\ &= \frac{\sin \theta \cdot \cos^2 \theta + \sin \theta}{\cos \theta \cdot \sin^2 \theta} \quad [\text{Since, } \sin^2 A + \cos^2 A = 1] \\ &= \frac{\sin \theta \cdot \cos^2 \theta}{\cos \theta \cdot \sin^2 \theta} + \frac{\sin \theta}{\cos \theta \cdot \sin^2 \theta} \\ &= \frac{\cos \theta}{\sin \theta} + \frac{1}{\cos \theta \cdot \sin \theta} \\ &= \cot \theta + \sec \theta \cdot \operatorname{cosec} \theta \\ &= \sec \theta \cdot \operatorname{cosec} \theta + \cot \theta \\ &= \text{RHS.} \end{aligned}$$

Hence, Proved.

25. We have,

$$\begin{aligned} \text{LHS} &= \frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} \\ \Rightarrow \text{LHS} &= \frac{(\tan \theta + \sec \theta) - 1}{(\tan \theta - \sec \theta) + 1} \\ \Rightarrow \text{LHS} &= \frac{(\sec \theta + \tan \theta) - (\sec^2 \theta - \tan^2 \theta)}{\tan \theta - \sec \theta + 1} \quad [\because \sec^2 \theta - \tan^2 \theta = 1] \\ \Rightarrow \text{LHS} &= \frac{(\sec \theta + \tan \theta) - (\sec \theta + \tan \theta)(\sec \theta - \tan \theta)}{\tan \theta - \sec \theta + 1} \end{aligned}$$

$$\begin{aligned} \Rightarrow \text{LHS} &= \frac{(\sec \theta + \tan \theta)[1 - (\sec \theta - \tan \theta)]}{\tan \theta - \sec \theta + 1} \\ \Rightarrow \text{LHS} &= \frac{(\sec \theta + \tan \theta)(1 - \sec \theta + \tan \theta)}{(\tan \theta - \sec \theta + 1)} \\ \Rightarrow \text{LHS} &= \frac{(\sec \theta + \tan \theta)(\tan \theta - \sec \theta + 1)}{(\tan \theta - \sec \theta + 1)} \\ \Rightarrow \text{LHS} &= \sec \theta + \tan \theta = \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} = \frac{1 + \sin \theta}{\cos \theta} = \text{RHS} \end{aligned}$$

$$\begin{aligned} 26. &= (m + n)^{2/3} + (m - n)^{2/3} \\ &= (a \cos^3 \theta + 3a \cos \theta \sin^2 \theta + a \sin^3 \theta + 3a \cos^2 \theta \sin \theta)^{2/3} + (a \cos^3 \theta + 3a \cos \theta \sin^2 \theta - a \sin^3 \theta - 3a \cos^2 \theta \sin \theta)^{2/3} \\ &= a^{2/3}(\cos^3 \theta + 3 \cos \theta \sin^2 \theta + \sin^3 \theta + 3 \cos^2 \theta \sin \theta)^{2/3} + a^{2/3}(\cos^3 \theta + 3 \cos \theta \sin^2 \theta - \sin^3 \theta - 3 \cos^2 \theta \sin \theta)^{2/3} \\ &= a^{2/3}\{(\cos \theta + \sin \theta)^3\}^{2/3} + a^{2/3}\{(\cos \theta - \sin \theta)^3\}^{2/3} \\ &= a^{2/3}(\cos \theta + \sin \theta)^2 + a^{2/3}(\cos \theta - \sin \theta)^2 \\ &= a^{2/3}(\cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta) + a^{2/3}(\cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta) \\ &= a^{2/3}(1 + 2 \cos \theta \sin \theta) + a^{2/3}(1 - 2 \cos \theta \sin \theta) \quad [\because \cos^2 \theta + \sin^2 \theta = 1] \\ &= a^{2/3}(1 + 2 \cos \theta \sin \theta + 1 - 2 \cos \theta \sin \theta) \\ &= a^{2/3}(1 + 1) \\ &= 2a^{2/3} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

27. We have,

$$\begin{aligned} \text{LHS} &= (\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2 \\ \Rightarrow \text{LHS} &= (\tan^2 A + \operatorname{cosec}^2 B + 2 \tan A \operatorname{cosec} B) - (\cot^2 B + \sec^2 A - 2 \cot B \sec A) \\ \Rightarrow \text{LHS} &= (\tan^2 A - \sec^2 A) + (\operatorname{cosec}^2 B - \cot^2 B) + 2 \tan A \operatorname{cosec} B + 2 \cot B \sec A \\ \text{But, } \sec^2 A - \tan^2 A &= 1 \text{ \& } \operatorname{cosec}^2 B - \cot^2 B = 1 \\ \therefore \text{LHS} &= -1 + 1 + 2 \tan A \operatorname{cosec} B + 2 \cot B \sec A \\ \Rightarrow \text{LHS} &= 2 (\tan A \operatorname{cosec} B + \cot B \sec A) \\ \Rightarrow \text{LHS} &= 2 \tan A \cot B \left(\frac{\operatorname{cosec} B}{\cot B} + \frac{\sec A}{\tan A} \right) \quad [\text{Dividing and multiplying by } \tan A \cot B] \\ \Rightarrow \text{LHS} &= 2 \tan A \cot B \left\{ \frac{\frac{1}{\sin B}}{\frac{\cos B}{\sin B}} + \frac{\frac{1}{\cos A}}{\frac{\sin A}{\cos A}} \right\} \quad [\text{Since, } \operatorname{Cosec} A \cdot \sin A = 1, \sec A \cdot \cos A = 1, (\sin A / \cos A) = \tan A \text{ \& } (\cos A / \sin A) = \cot A] \\ \Rightarrow \text{LHS} &= 2 \tan A \cot B \left(\frac{1}{\cos B} + \frac{1}{\sin A} \right) = 2 \tan A \cot B (\sec B + \operatorname{cosec} A) = \text{RHS. Hence, proved.} \end{aligned}$$

28. LHS

$$\begin{aligned} &= \frac{1}{\cot^2 \theta} + \frac{1}{1 + \tan^2 \theta} \\ &= \tan^2 \theta + \frac{1}{\sec^2 \theta} \\ &= \tan^2 \theta + \cos^2 \theta \\ &= (\sec^2 \theta - 1) + \cos^2 \theta \\ &= \sec^2 \theta - (1 - \cos^2 \theta) \\ &= \sec^2 \theta - \sin^2 \theta \\ &= \frac{1}{\cos^2 \theta} - \sin^2 \theta \\ &= \frac{1}{1 - \sin^2 \theta} - \frac{1}{\operatorname{cosec}^2 \theta} \\ &= \text{R.H.S} \end{aligned}$$

Hence proved.

$$\begin{aligned} 29. \text{L.H.S: } &1 + \frac{\cot^2 \alpha}{1 + \operatorname{cosec} \alpha} \\ &= 1 + \frac{\cos^2 \alpha / \sin^2 \alpha}{1 + 1 / \sin \alpha} \\ &[\because \cot^2 \alpha = \frac{\cos^2 \alpha}{\sin^2 \alpha} \text{ and } \operatorname{cosec} \alpha = \frac{1}{\sin \alpha}] \\ &[\because \cot^2 \alpha = \frac{\cos^2 \alpha}{\sin^2 \alpha} \text{ and } \operatorname{cosec} \alpha = \frac{1}{\sin \alpha}] \\ &= 1 + \frac{\cos^2 \alpha / \sin^2 \alpha}{\frac{\sin \alpha + 1}{\sin \alpha}} \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{\cos^2 \alpha}{\sin \alpha(1+\sin \alpha)} \\
&= \frac{\sin \alpha + \sin^2 \alpha + \cos^2 \alpha}{\sin \alpha(1+\sin \alpha)} \\
&= \frac{1+\sin \alpha}{\sin \alpha(1+\sin \alpha)} \left[\because \sin^2 \alpha + \cos^2 \alpha = 1 \right] \\
&= \frac{1}{\sin \alpha} = \operatorname{cosec} \alpha \left[\because \frac{1}{\sin \alpha} = \operatorname{cosec} \alpha \right]
\end{aligned}$$

30. Here:

$$\begin{aligned}
L.H.S &= (\sin \theta + \cos \theta + 1)(\sin \theta + \cos \theta - 1) \operatorname{sec} \theta \operatorname{cosec} \theta \\
&= [(\sin \theta + \cos \theta)^2 - 1] \frac{1}{\cos \theta \sin \theta} \\
&= [(\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta) - 1] \frac{1}{\cos \theta \sin \theta} \\
&= [1 + 2 \sin \theta \cos \theta - 1] \frac{1}{\cos \theta \sin \theta} \\
&= 2 \sin \theta \cos \theta \times \frac{1}{\cos \theta \sin \theta} \\
&= 2 \\
&= R.H.S
\end{aligned}$$

31. $(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \lambda) = (1 - \cos \alpha)(1 - \cos \beta)(1 - \cos \lambda)$

Solving L.H.S.

$$(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \lambda)$$

multiplying and dividing by $\sin \alpha \sin \beta \sin \lambda$ we get,

$$\frac{(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \lambda) \sin \alpha \sin \beta \sin \lambda}{\sin \alpha \sin \beta \sin \lambda} \sin \alpha \sin \beta \sin \lambda$$

$$\text{Now, } \frac{1 + \cos \alpha}{\sin \alpha} = \frac{2 \cos^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \cot \frac{\alpha}{2}$$

$$\text{Similarly, } \frac{1 + \cos \beta}{\sin \beta} = \cot \frac{\beta}{2}, \frac{1 + \cos \lambda}{\sin \lambda} = \cot \frac{\lambda}{2}$$

$$\cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\lambda}{2} \sin \alpha \sin \beta \sin \lambda$$

Solving R.H.S.

$$(1 - \cos \alpha)(1 - \cos \beta)(1 - \cos \lambda)$$

$$= 2 \sin^2 \frac{\alpha}{2} \times 2 \sin^2 \frac{\beta}{2} \times 2 \sin^2 \frac{\lambda}{2}$$

$$= 2 \sin \frac{\alpha}{2} \sin \frac{\alpha}{2} \times 2 \sin \frac{\beta}{2} \sin \frac{\beta}{2} \times 2 \sin \frac{\lambda}{2} \sin \frac{\lambda}{2}$$

Multiplying and dividing by $\cos \frac{\alpha}{2} \times \cos \frac{\beta}{2} \times \cos \frac{\lambda}{2}$ we get

$$= 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \times 2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} \times 2 \sin \frac{\lambda}{2} \cos \frac{\lambda}{2} \times \frac{\sin \frac{\alpha}{2} \times \sin \frac{\beta}{2} \times \sin \frac{\lambda}{2}}{\cos \frac{\alpha}{2} \times \cos \frac{\beta}{2} \times \cos \frac{\lambda}{2}}$$

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\lambda}{2} \sin \alpha \sin \beta \sin \lambda$$

Hence, $\sin \alpha \sin \beta \sin \lambda$ is member of this equality.

$$\begin{aligned}
32. \text{ LHS} &= \frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} + \frac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} \\
&= \frac{(\cos \theta + \sin \theta)(\cos^2 \theta + \sin^2 \theta - \sin \theta \cos \theta)}{(\cos \theta + \sin \theta)} + \frac{(\cos \theta - \sin \theta)(\cos^2 \theta + \sin^2 \theta + \sin \theta \cos \theta)}{(\cos \theta - \sin \theta)}
\end{aligned}$$

$$= (1 - \sin \theta \cos \theta) + (1 + \sin \theta \cos \theta)$$

$$= 1 + 1 - \sin \theta \cos \theta + \sin \theta \cos \theta$$

$$= 2 = \text{RHS}$$

33. Given,

$$\sin \theta + \cos \theta = p \dots\dots(1)$$

$$\text{And, } \sec \theta + \operatorname{cosec} \theta = q \dots\dots(2)$$

Now, L.H.S

$$= q(p^2 - 1)$$

$$= (\sec \theta + \operatorname{cosec} \theta) [(\sin \theta + \cos \theta)^2 - 1] \quad \text{【 from (1) \& (2) 】}$$

$$= \left[\frac{1}{\cos \theta} + \frac{1}{\sin \theta} \right] [\sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta - 1]$$

$$= \left[\frac{\sin \theta + \cos \theta}{\cos \theta \sin \theta} \right] [1 + 2 \cos \theta \sin \theta - 1] \quad (\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$= \frac{\sin \theta + \cos \theta}{\cos \theta \sin \theta} \times 2 \cos \theta \sin \theta$$

$$= 2(\sin \theta + \cos \theta)$$

$$= 2p (\because \sin \theta + \cos \theta = p)$$

=R.H.S

Hence, proved.

34. Given,

$$\tan \theta + \sin \theta = m \dots (1)$$

$$\&, \tan \theta - \sin \theta = n \dots (2)$$

$$\text{Now, LHS} = m^2 - n^2$$

$$= (\tan \theta + \sin \theta)^2 - (\tan \theta - \sin \theta)^2 \text{ [from (1) \& (2)]}$$

$$= \tan^2 \theta + \sin^2 \theta + 2 \tan \theta \sin \theta - \tan^2 \theta - \sin^2 \theta + 2 \tan \theta \sin \theta$$

$$= 4 \tan \theta \sin \theta$$

$$= 4 \sqrt{\tan^2 \theta \cdot \sin^2 \theta}$$

$$= 4 \sqrt{\tan^2 \theta (1 - \cos^2 \theta)}$$

$$= 4 \sqrt{\tan^2 \theta - \sin^2 \theta}$$

$$= 4 \sqrt{(\tan \theta + \sin \theta)(\tan \theta - \sin \theta)}$$

$$= 4 \sqrt{mn} \text{ [from (1) \& (2)]}$$

= RHS. Hence, Proved.

35. Given, $m = \frac{\cos \alpha}{\cos \beta} \dots (1)$ and, $n = \frac{\cos \alpha}{\sin \beta} \dots (2)$

$$\text{LHS} = (m^2 + n^2) \cos^2 \beta$$

$$\Rightarrow \text{LHS} = \left(\frac{\cos^2 \alpha}{\cos^2 \beta} + \frac{\cos^2 \alpha}{\sin^2 \beta} \right) \cos^2 \beta \text{ [from (1) \& (2)]}$$

$$\Rightarrow \text{LHS} = \left(\frac{\cos^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta}{\cos^2 \beta \sin^2 \beta} \right) \cos^2 \beta$$

$$\Rightarrow \text{LHS} = \cos^2 \alpha \left(\frac{\sin^2 \beta + \cos^2 \beta}{\cos^2 \beta \sin^2 \beta} \right) \cos^2 \beta$$

$$\Rightarrow \text{LHS} = \cos^2 \alpha \left(\frac{1}{\cos^2 \beta \sin^2 \beta} \right) \cos^2 \beta \text{ [Since, } \sin^2 A + \cos^2 A = 1 \text{]}$$

$$\Rightarrow \text{L.H.S.} = \frac{\cos^2 \alpha}{\sin^2 \beta} = \left(\frac{\cos \alpha}{\sin \beta} \right)^2$$

$$\therefore \text{L.H.S.} = n^2 \text{ [from (2)]}$$

= R.H.S . Hence, Proved.

36. Given- $\sin^6 A + 3 \sin^2 A \cos^2 A = 1 - \cos^6 A$

Now, taking

$$\sin^6 A + \cos^6 A = 1 - 3 \sin^2 A \cos^2 A$$

Taking LHS

$$= \sin^6 A + \cos^6 A = (\sin^2 A)^3 + (\cos^2 A)^3$$

$$= (\sin^2 A + \cos^2 A)^3 - 3 \sin^2 A \cos^2 A (\sin^2 A + \cos^2 A) \{ \because a^3 + b^3 = (a + b)^3 - 3ab(a + b) \}$$

$$= (1)^3 - 3 \sin^2 A \cos^2 A (1)$$

$$= 1 - 3 \sin^2 A \cos^2 A = \text{RHS}$$

37. First, we will show that, $\frac{1 + \tan^2 A}{1 + \cot^2 A} = \tan^2 A$

$$\text{LHS} = \frac{1 + \tan^2 A}{1 + \cot^2 A} = \frac{1 + \tan^2 A}{1 + \frac{1}{\tan^2 A}} = \frac{1 + \tan^2 A}{\frac{1 + \tan^2 A}{\tan^2 A}}$$

$$= (1 + \tan^2 A) \times \frac{\tan^2 A}{1 + \tan^2 A}$$

$$= \tan^2 A = \text{RHS} \dots (i)$$

Now, we will show that, $\left(\frac{1 - \tan A}{1 - \cot A} \right)^2 = \tan^2 A$

$$\text{LHS} = \left(\frac{1 - \tan A}{1 - \cot A} \right)^2 = \left(\frac{1 - \tan A}{1 - \frac{1}{\tan A}} \right)^2 = \left(\frac{1 - \tan A}{\frac{\tan A - 1}{\tan A}} \right)^2$$

$$= \left[(1 - \tan A) \times \left(\frac{\tan A}{-(1 - \tan A)} \right) \right]^2$$

$$= (\tan A)^2 = \tan^2 A = \text{RHS} \dots (ii)$$

Hence, from (i) and (ii),

$$\frac{1+\tan^2 A}{1+\cot^2 A} = \left(\frac{1-\tan A}{1-\cot A} \right)^2 = \tan^2 A$$

Hence proved.

38. $\sin A$ can be expressed in terms of $\sec A$ as:

$$\sin A = \sqrt{\sin^2 A}$$

$$\sin A = \sqrt{(1 - \cos^2 A)}$$

$$\sin A = \sqrt{1 - \frac{1}{\sec^2 A}}$$

$$\sin A = \sqrt{\frac{\sec^2 A - 1}{\sec^2 A}}$$

$$\sin A = \frac{1}{\sec A} \sqrt{\sec^2 A - 1}$$

Now,

$\cos A$ can be expressed in terms of $\sec A$ as:

$$\cos A = \frac{1}{\sec A}$$

$\tan A$ can be expressed in the form of $\sec A$ as:

$$\text{As, } 1 + \tan^2 A = \sec^2 A$$

$$\Rightarrow \tan A = \pm \sqrt{(\sec^2 A - 1)}$$

since A is an acute angle, and $\tan A$ is positive when A is acute, So, $\tan A = \sqrt{(\sec^2 A - 1)}$

Now $\operatorname{cosec} A$ can be expressed in the form of $\sec A$ as:

$$\operatorname{cosec} A = \frac{1}{\sin A}$$

$$\operatorname{cosec} A = \frac{1}{\frac{1}{\frac{\sec A}{\sqrt{1-\sec^2 A}}}}}$$

$$\operatorname{cosec} A = \frac{\sqrt{1-\sec^2 A}}{\sec A}$$

Now, $\cot A$ can be expressed in terms of $\sec A$ as:

$$\cot A = \frac{1}{\tan A}$$

$$\text{as, } 1 + \tan^2 A = \sec^2 A$$

$$\cot A = \frac{1}{\sqrt{\sec^2 A - 1}}$$

39. LHS = $\sec \theta (1 - \sin \theta)(\sec \theta + \tan \theta)$

$$= \left[\sec \theta - \frac{\sin \theta}{\cos \theta} \right] \times (\sec \theta + \tan \theta)$$

$$= (\sec \theta - \tan \theta)(\sec \theta + \tan \theta)$$

$$= \sec^2 \theta - \tan^2 \theta = 1 = \text{RHS}$$

40. LHS = $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$

$$= \sin^2 A + \operatorname{cosec}^2 A + 2 \sin A \operatorname{cosec} A + \cos^2 A + \sec^2 A + 2 \cos A \sec A$$

$$= \sin^2 A + \cos^2 A + \operatorname{cosec}^2 A + \sec^2 A + 2 + 2$$

$$= 1 + \operatorname{cosec}^2 A + \sec^2 A + 4$$

$$= (1 + \cot^2 A) + (1 + \tan^2 A) + 5$$

$$= 7 + \tan^2 A + \cot^2 A = \text{RHS}$$

41. We have to express the trigonometric ratios $\sin A$, $\sec A$ and $\tan A$ in terms of $\cot A$.

For $\sin A$,

By using identity $\operatorname{cosec}^2 A - \cot^2 A = 1$

$$\Rightarrow \operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\Rightarrow \frac{1}{\sin^2 A} = 1 + \cot^2 A$$

$$\Rightarrow \sin^2 A = \frac{1}{1 + \cot^2 A}$$

$$\Rightarrow \sin A = \frac{1}{\sqrt{1 + \cot^2 A}}$$

For $\sec A$,

By using identity $\sec^2 A - \tan^2 A = 1$

$$\Rightarrow \sec^2 A = 1 + \tan^2 A$$

$$\Rightarrow \sec^2 A = 1 + \frac{1}{\cot^2 A} = \frac{\cot^2 A + 1}{\cot^2 A}$$

$$\Rightarrow \sec^2 A = \frac{1 + \cot^2 A}{\cot^2 A}$$

$$\Rightarrow \sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}$$

For $\tan A$,

$$\tan A = \frac{1}{\cot A}$$

42. We have;

$$\begin{aligned} & (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 \\ &= 4 \sin^2 \theta + 9 \cos^2 \theta + 12 \sin \theta \cos \theta + 9 \sin^2 \theta + 4 \cos^2 \theta - 12 \sin \theta \cos \theta \\ &= 13 \sin^2 \theta + 13 \cos^2 \theta \\ &= 13 (\sin^2 \theta + \cos^2 \theta) \\ &= 13 \end{aligned}$$

$$\text{Now, } (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 = 13$$

$$\Rightarrow (2)^2 + (3 \sin \theta - 2 \cos \theta)^2 = 13$$

$$\Rightarrow 4 + (3 \sin \theta - 2 \cos \theta)^2 = 13$$

$$\Rightarrow (3 \sin \theta - 2 \cos \theta)^2 = 9$$

$$\Rightarrow 3 \sin \theta - 2 \cos \theta = \pm 3$$

43. Take ,

$$\text{LHS} = 2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1$$

$$\Rightarrow \text{LHS} = 2 \{(\sin^2 \theta)^3 + (\cos^2 \theta)^3\} - 3 \{(\sin^2 \theta)^2 + (\cos^2 \theta)^2\} + 1$$

Using $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ and $a^2 + b^2 = (a + b)^2 - 2ab$ in above expression, where $a = \sin^2 \theta$ & $b = \cos^2 \theta$; we get :-

$$\text{LHS} = 2 \{(\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)\} - 3 \{(\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta\} + 1$$

$$\Rightarrow \text{LHS} = 2(1 - 3 \sin^2 \theta \cos^2 \theta) - 3(1 - 2 \sin^2 \theta \cos^2 \theta) + 1 \text{ [Since, } \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \text{LHS} = 2 - 6 \sin^2 \theta \cos^2 \theta - 3 + 6 \sin^2 \theta \cos^2 \theta + 1$$

$$\text{Hence, L.H.S.} = 0 = \text{R.H.S.}$$

Hence, proved.

44. L.H.S.

$$= \sec A(1 - \sin A)(\sec A + \tan A)$$

$$= \frac{1}{\cos A}(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)}{\cos A} \left(\frac{1 + \sin A}{\cos A}\right)$$

$$= \frac{\cos A}{(1 - \sin A)(1 + \sin A)}$$

$$= \frac{\cos A \times \cos A}{(1^2 - \sin^2 A)} \text{ [Since, } (a - b)(a + b) = a^2 - b^2 \text{]}$$

$$= \frac{\cos^2 A}{(1 - \sin^2 A)}$$

$$= \frac{\cos^2 A}{\cos^2 A}$$

$$= 1$$

$$= \text{RHS}$$

Hence, proved.

45. We have to prove that :-

$$\Rightarrow \sec^6 \theta = \tan^6 \theta + 3 \tan^2 \theta \sec^2 \theta + 1$$

$$\Rightarrow \sec^6 \theta - \tan^6 \theta = 3 \tan^2 \theta \sec^2 \theta + 1$$

Now, LHS

$$= \sec^6 \theta - \tan^6 \theta$$

$$= (\sec^2 \theta)^3 - (\tan^2 \theta)^3$$

$$= (\sec^2 \theta - \tan^2 \theta) \left[(\sec^2 \theta)^2 + \sec^2 \theta \tan^2 \theta + (\tan^2 \theta)^2 \right] \text{ {Since, } a^3 - b^3 = (a - b)(a^2 - ab + b^2) \text{}}$$

$$= 1 \left[\sec^4 \theta + \sec^2 \theta \tan^2 \theta + \tan^4 \theta \right] \text{ [} \because \sec^2 \theta - \tan^2 \theta = 1 \text{]}$$

$$= \sec^4 \theta + \tan^4 \theta + \sec^2 \theta \tan^2 \theta$$

$$= (\sec^2 \theta)^2 + (\tan^2 \theta)^2 + \sec^2 \theta \tan^2 \theta$$

$$\begin{aligned}
& \text{Adding and subtracting } 2\sec^2\theta\tan^2\theta \\
& = (\sec^2\theta)^2 + (\tan^2\theta)^2 - 2\sec^2\theta\tan^2\theta + 2\sec^2\theta\tan^2\theta + \sec^2\theta\tan^2\theta \\
& = (\sec^2\theta - \tan^2\theta)^2 + 3\sec^2\theta\tan^2\theta \left[\because (a-b)^2 = a^2 + b^2 - 2ab \right] \\
& = 1 + 3\sec^2\theta\tan^2\theta \left[\because \sec^2\theta - \tan^2\theta = 1 \right] \\
& = \text{RHS}
\end{aligned}$$

Hence proved

46. Given that, $\sin \theta + \cos \theta = \sqrt{2}$

On squaring both the sides, we get

$$\begin{aligned}
(\sin \theta + \cos \theta)^2 & = (\sqrt{2})^2 \\
\sin^2\theta + \cos^2\theta + 2 \sin \theta \cos \theta & = 2 \left[\because (a+b)^2 = a^2 + 2ab + b^2 \right] \\
\Rightarrow 1 + 2 \sin \theta \cos \theta & = 2 \\
\Rightarrow 2 \sin \theta \cos \theta & = 2 - 1 = 1 \\
\Rightarrow \frac{1}{\sin \theta \cos \theta} & = 2 \dots\dots\dots(i)
\end{aligned}$$

Now, $\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$

$$\frac{\sin^2\theta + \cos^2\theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} \dots\dots\dots(ii)$$

From (i) and (ii) we get

$$\tan \theta + \cot \theta = 2$$

47. $(\tan \theta + \sin \theta) = m$ and $(\tan \theta - \sin \theta) = n$

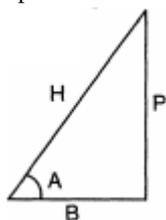
$$\begin{aligned}
\text{LHS} & = (m^2 - n^2)^2 \\
& = [(\tan \theta + \sin \theta)^2 - (\tan \theta - \sin \theta)^2]^2 \\
& = [4 \tan \theta \sin \theta]^2 \left[\because (a+b)^2 - (a-b)^2 = 4ab \right] \\
& = 16 \tan^2 \theta \sin^2 \theta \dots(1)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} & = 16mn = 16(\tan \theta + \sin \theta)(\tan \theta - \sin \theta) \\
& = 16(\tan^2 \theta - \sin^2 \theta) = 16 \left(\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \right) \\
& = 16 \left(\frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \right) \\
& = 16 \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} \left[\because 1 - \cos^2 \theta = \sin^2 \theta \right] \\
& = 16 \frac{\sin^2 \theta}{\cos^2 \theta} \times \sin^2 \theta
\end{aligned}$$

$$\text{RHS} = 16 \tan^2 \sin^2 \theta \dots(2)$$

\therefore LHS = RHS

48. If two expressions are equal for all the values of same parameter or parameters, then the statement of equality between the two expressions is called an identity.



Let, $\tan A = \frac{P}{B}$, $\sec A = \frac{H}{B}$

$$H^2 = P^2 + B^2$$

$$\text{LHS} = 1 + \tan^2 A$$

$$\begin{aligned}
& = 1 + \left(\frac{P}{B} \right)^2 = 1 + \frac{P^2}{B^2} \\
& = \frac{B^2 + P^2}{B^2} = \frac{H^2}{B^2} \\
& = \left(\frac{H}{B} \right)^2
\end{aligned}$$

$$= \sec^2 A$$

= R.H.S

Hence Proved.

49. $\text{LHS} = \left(\frac{1}{\sin A} - \sin A \right) \left(\frac{1}{\cos A} - \cos A \right)$
 $= \frac{1 - \sin^2 A}{\sin A} \times \frac{1 - \cos^2 A}{\cos A}$

$$\begin{aligned}
&= \sin A \cos A \\
\text{RHS} &= \frac{1}{\frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}} \\
&= \frac{\sin A \cos A}{\sin^2 A + \cos^2 A} \\
&= \sin A \cos A = \text{LHS}
\end{aligned}$$

$$\begin{aligned}
50. \text{LHS} &= (1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) \\
&= (1 + \tan^2 A) + \frac{(\tan^2 A + 1)}{\tan^2 A} \\
&= \sec^2 A + \frac{\sec^2 A}{\tan^2 A} \left[\because 1 + \tan^2 A = \sec^2 A \right] \\
&= \frac{1}{\cos^2 A} + \frac{\frac{1}{\cos^2 A}}{\frac{\sin^2 A}{\cos^2 A}} \left[\begin{array}{l} \because \sec^2 A = \frac{1}{\cos^2 A} \\ \tan^2 A = \frac{\sin^2 A}{\cos^2 A} \end{array} \right] \\
&= \frac{1}{\cos^2 A} + \frac{1}{\cos^2 A} \times \frac{\cos^2 A}{\sin^2 A} \\
&= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} \\
&= \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A} \\
&= \frac{\cos^2 A \sin^2 A}{\cos^2 A \sin^2 A} \left[\because \sin^2 A + \cos^2 A = 1 \right] \\
&= \frac{1}{(1 - \sin^2 A) \sin^2 A} \left[\because \cos^2 A = 1 - \sin^2 A \right] \\
&= \frac{1}{\sin^2 A - \sin^4 A} \\
&= \text{RHS}
\end{aligned}$$

Hence proved.

51. We have to prove that, $\frac{\sin \theta - \cos \theta + 1}{\sin \theta + \cos \theta - 1} = \frac{1}{\sec \theta - \tan \theta}$ using identity $\sec^2 \theta = 1 + \tan^2 \theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sin \theta - \cos \theta + 1}{\sin \theta + \cos \theta - 1} = \frac{\tan \theta - 1 + \sec \theta}{\tan \theta + 1 - \sec \theta} \left[\text{dividing the numerator and denominator by } \cos \theta \right] \\
&= \frac{(\tan \theta + \sec \theta) - 1}{(\tan \theta - \sec \theta) + 1} = \frac{\{(\tan \theta + \sec \theta - 1 \} (\tan \theta - \sec \theta)}{\{(\tan \theta - \sec \theta + 1 \} (\tan \theta - \sec \theta)} \left[\text{Multiplying and dividing by } (\tan \theta - \sec \theta) \right] \\
&= \frac{(\tan^2 \theta - \sec^2 \theta) - (\tan \theta - \sec \theta)}{\{(\tan \theta - \sec \theta + 1 \} (\tan \theta - \sec \theta)} \left[\because (a - b)(a + b) = a^2 - b^2 \right] \\
&= \frac{-1 - \tan \theta + \sec \theta}{(\tan \theta - \sec \theta + 1)(\tan \theta - \sec \theta)} \left[\because \tan^2 \theta - \sec^2 \theta = -1 \right] \\
&= \frac{-(\tan \theta - \sec \theta + 1)}{(\tan \theta - \sec \theta + 1)(\tan \theta - \sec \theta)} = \frac{-1}{\tan \theta - \sec \theta} \\
&= \frac{1}{\sec \theta - \tan \theta} = \text{RHS}
\end{aligned}$$

Hence Proved.

52. We have,

$$\begin{aligned}
\text{LHS} &= \frac{p^2 - 1}{p^2 + 1} = \frac{(\sec \theta + \tan \theta)^2 - 1}{(\sec \theta + \tan \theta)^2 + 1} \\
\Rightarrow \text{LHS} &= \frac{\sec^2 \theta + \tan^2 \theta + 2 \sec \theta \tan \theta - 1}{\sec^2 \theta + \tan^2 \theta + 2 \sec \theta \tan \theta + 1} \\
\Rightarrow \text{LHS} &= \frac{(\sec^2 \theta - 1) + \tan^2 \theta + 2 \sec \theta \tan \theta}{\sec^2 \theta + 2 \sec \theta \tan \theta + (1 + \tan^2 \theta)} \\
\Rightarrow \text{LHS} &= \frac{\tan^2 \theta + \tan^2 \theta + 2 \sec \theta \tan \theta}{\sec^2 \theta + 2 \sec \theta \tan \theta + \sec^2 \theta} \\
\Rightarrow \text{LHS} &= \frac{2 \tan^2 \theta + 2 \tan \theta \sec \theta}{2 \sec^2 \theta + 2 \sec \theta \tan \theta} = \frac{2 \tan \theta (\tan \theta + \sec \theta)}{2 \sec \theta (\sec \theta + \tan \theta)} = \frac{\tan \theta}{\sec \theta} = \frac{\sin \theta}{\cos \theta \cdot \sec \theta} = \sin \theta = \text{RHS}
\end{aligned}$$

53. Given, $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$, then we have to prove that $\tan \theta = 1$, or $\frac{1}{2}$.

$$\text{Now, } 1 + \sin^2 \theta = 3 \sin \theta \cos \theta$$

[Dividing by $\sin^2 \theta$ On both sides]

$$\begin{aligned}
\Rightarrow \frac{1}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} &= \frac{3 \sin \theta \cos \theta}{\sin^2 \theta} \\
\Rightarrow \text{cosec}^2 \theta + 1 &= 3 \cot \theta \\
\Rightarrow 1 + \cot^2 \theta + 1 - 3 \cot \theta &= 0 \\
\Rightarrow \cot^2 \theta - 3 \cot \theta + 2 &= 0 \\
\Rightarrow \cot^2 \theta - 2 \cot \theta - \cot \theta + 2 &= 0 \\
\Rightarrow \cot \theta (\cot \theta - 2) - 1(\cot \theta - 2) &= 0 \\
\Rightarrow (\cot \theta - 2)(\cot \theta - 1) &= 0 \\
\Rightarrow \cot \theta - 2 = 0 \text{ or } (\cot \theta - 1) &= 0
\end{aligned}$$

$$\Rightarrow \cot\theta = 2 \text{ or } \cot\theta = 1$$

$$\Rightarrow \tan\theta = \frac{1}{2} \text{ or } \tan\theta = 1$$

Hence, either, $\tan\theta = \frac{1}{2}$, or 1

54. Given, $\sec\theta + \tan\theta = p$

$$\Rightarrow \sec\theta = p - \tan\theta \dots\dots\dots(1)$$

Now,

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$$

$$\Rightarrow p(\sec\theta - \tan\theta) = 1$$

$$\Rightarrow \sec\theta - \tan\theta = \frac{1}{p} \dots\dots\dots(2)$$

From (1) and (2),

$$p - \tan\theta - \tan\theta = \frac{1}{p}$$

$$\Rightarrow p - 2\tan\theta = \frac{1}{p}$$

$$\Rightarrow 2\tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \tan\theta = \frac{1}{2} \left(p - \frac{1}{p} \right)$$

55. We have, $1 + \sin^2\theta = 3\sin\theta \cos\theta$

Dividing both sides by $\cos^2\theta$, we obtain

$$\frac{1 + \sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta \cos\theta}{\cos^2\theta}$$

$$\Rightarrow \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta}{\cos\theta}$$

$$\Rightarrow \sec^2\theta + \tan^2\theta = 3\tan\theta$$

$$\Rightarrow 1 + \tan^2\theta + \tan^2\theta = 3\tan\theta \text{ [}\because \sec^2\theta = 1 + \tan^2\theta \text{]}$$

$$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$$

$$\Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$$

$$\Rightarrow 2\tan\theta(\tan\theta - 1) - (\tan\theta - 1) = 0$$

$$\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) = 0$$

$$\Rightarrow 2\tan\theta - 1 = 0 \text{ or } \tan\theta - 1 = 0 \Rightarrow 2\tan\theta = 1 \text{ or } \tan\theta = 1 \Rightarrow \tan\theta = \frac{1}{2} \text{ or } \tan\theta = 1$$

56. Given: $\operatorname{cosec} A + \cot A = m$

$$\Rightarrow (\operatorname{cosec} A + \cot A)^2 = (m)^2 \text{ [squaring both sides]}$$

$$\Rightarrow \operatorname{cosec}^2 A + \cot^2 A + 2 \operatorname{cosec} A \cot A = m^2 \dots\dots\dots(1)$$

Now, LHS

$$= \frac{m^2 - 1}{m^2 + 1}$$

$$= \frac{\operatorname{cosec}^2 A + \cot^2 A + 2 \operatorname{cosec} A \cot A - 1}{\operatorname{cosec}^2 A + \cot^2 A + 2 \operatorname{cosec} A \cot A + 1} \text{ [From (1)]}$$

$$= \frac{\cot^2 A + \cot^2 A + 2 \operatorname{cosec} A \cot A}{\operatorname{cosec}^2 A + \operatorname{cosec}^2 A + 2 \operatorname{cosec} A \cot A} \text{ [Since, } \operatorname{Cosec}^2 A - \cot^2 A = 1 \text{]}$$

$$= \frac{2 \cot^2 A + 2 \operatorname{cosec} A \cot A}{2 \operatorname{cosec}^2 A + 2 \operatorname{cosec} A \cot A}$$

$$= \frac{2 \cot A (\cot A + \operatorname{cosec} A)}{2 \operatorname{cosec} A (\operatorname{cosec} A + \cot A)}$$

$$= \frac{\cot A}{\operatorname{cosec} A}$$

$$= \frac{\cos A}{\sin A}$$

$$= \frac{1}{\frac{\sin A}{\cos A}}$$

$$= \frac{\cos A}{\sin A} \times \frac{\sin A}{1}$$

$$= \cos A = \text{RHS}$$

Hence, Proved.

$$57. \text{L.H.S} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \sqrt{\frac{1 - \cos 60^\circ}{1 + \cos 60^\circ}}$$

$$= \sqrt{\frac{1 - \frac{1}{2}}{1 + \frac{1}{2}}} \quad (\because \cos 60^\circ = \frac{1}{2})$$

$$= \sqrt{\frac{\frac{1}{2}}{\frac{3}{2}}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\begin{aligned} \text{R.H.S} &= \frac{\sin \theta}{1+\cos \theta} = \frac{\sin 60^\circ}{1+\cos 60^\circ} \\ &= \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} \\ &= \frac{1}{\sqrt{3}} = \text{L.H.S} \end{aligned}$$

therefore, L.H.S = R.H.S

Hence, relation is verified for $\theta = 60^\circ$.

58. We have,

$$\begin{aligned} \text{LHS} &= (\text{cosec} \theta - \sin \theta) (\sec \theta - \cos \theta) \\ \Rightarrow \text{LHS} &= \left(\frac{1}{\sin \theta} - \sin \theta \right) \left(\frac{1}{\cos \theta} - \cos \theta \right) \\ \Rightarrow \text{LHS} &= \frac{1-\sin^2 \theta}{\sin \theta} \times \frac{1-\cos^2 \theta}{\cos \theta} \\ \Rightarrow \text{LHS} &= \frac{\cos^2 \theta}{\sin \theta} \times \frac{\sin^2 \theta}{\cos \theta} \quad (\text{since, } \sin^2 A + \cos^2 A = 1) \\ \Rightarrow \text{LHS} &= \frac{\sin \theta \cos \theta}{1} = \frac{\sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta} = \frac{1}{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}} \\ \Rightarrow \text{LHS} &= \frac{1}{\frac{\sin^2 \theta}{\sin \theta \cos \theta} + \frac{\cos^2 \theta}{\sin \theta \cos \theta}} = \frac{1}{\tan \theta + \cot \theta} = \text{RHS} \end{aligned}$$

59. Given ,

$$\begin{aligned} 2\sin^2 \theta - \cos^2 \theta &= 2 \\ \Rightarrow 2(1 - \cos^2 \theta) - \cos^2 \theta &= 2 \\ \Rightarrow 2 - 2\cos^2 \theta - \cos^2 \theta &= 2 \\ \Rightarrow 2 - 3\cos^2 \theta &= 2 \\ \Rightarrow 3\cos^2 \theta &= 0 \\ \Rightarrow \cos^2 \theta &= 0 \\ \Rightarrow \cos^2 \theta &= \cos^2 90^\circ \\ \Rightarrow \theta &= 90^\circ \end{aligned}$$

60. We have,

$$\begin{aligned} \text{LHS} &= 2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 \\ &= 2\left\{ (\sin^2 \theta)^3 + (\cos^2 \theta)^3 \right\} - 3(\sin^4 \theta + \cos^4 \theta) + 1 \end{aligned}$$

Using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ and $a^2 + b^2 = (a+b)^2 - 2ab$, we obtain

$$\begin{aligned} \text{LHS} &= 2\left\{ (\sin^2 \theta + \cos^2 \theta)^3 - 3\sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) \right\} - 3\left\{ (\sin^2 \theta + \cos^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta + 1 \right\} \\ &= 2(1 - 3\sin^2 \theta \cos^2 \theta) - 3(1 - 2\sin^2 \theta \cos^2 \theta) + 1 \\ &= 2 - 6\sin^2 \theta \cos^2 \theta - 3 + 6\sin^2 \theta \cos^2 \theta + 1 = 0 = \text{RHS} \end{aligned}$$

$$\begin{aligned} 61. \text{LHS} &= \frac{\cot^2 \theta (\sec \theta - 1)}{(1+\sin \theta)} + \frac{\sec^2 \theta (\sin \theta - 1)}{(1+\sec \theta)} \\ &= \frac{\cot^2 \theta (\sec \theta - 1)(1+\sec \theta) + \sec^2 \theta (\sin \theta - 1)(1+\sin \theta)}{(1+\sin \theta)(1+\sec \theta)} \\ &= \frac{\cot^2 \theta (\sec^2 \theta - 1) + \sec^2 \theta (\sin^2 \theta - 1)}{(1+\sin \theta)(1+\sec \theta)} \\ &= \frac{\cot^2 \theta \tan^2 \theta + \sec^2 \theta (-\cos^2 \theta)}{(1+\sin \theta)(1+\sec \theta)} \\ &= \frac{\cot^2 \theta \tan^2 \theta - \sec^2 \theta \cos^2 \theta}{(1+\sin \theta)(1+\sec \theta)} \\ &= \frac{\cot^2 \theta \times \frac{1}{\cot^2 \theta} - \sec^2 \theta \times \frac{1}{\sec^2 \theta}}{(1+\sin \theta)(1+\sec \theta)} \\ &= \frac{1-1}{(1+\sin \theta)(1+\sec \theta)} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

62. Given, $\sec \theta + \tan \theta = p \dots (1)$

$$\begin{aligned} \text{We know that, } \sec^2 \theta - \tan^2 \theta &= 1 \\ \Rightarrow (\sec \theta + \tan \theta) (\sec \theta - \tan \theta) &= 1 \\ \Rightarrow p(\sec \theta - \tan \theta) &= 1 \text{ from (1)} \\ \Rightarrow \sec \theta - \tan \theta &= \frac{1}{p} \dots (2) \end{aligned}$$

Adding (1) and (2), we get;

$$\Rightarrow (\sec \theta + \tan \theta) + (\sec \theta - \tan \theta) = p + \frac{1}{p}$$

and on subtracting (2) from (1), we get;

$$(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = p - \frac{1}{p}$$

$$\Rightarrow 2\tan\theta = p + \frac{1}{p}$$

$$\text{and, } 2 \tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right) \text{ and, } \tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right)$$

$$\text{Also, } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \sin\theta = \tan\theta \cdot \cos\theta$$

$$\therefore \sin\theta = \frac{\tan\theta}{\sec\theta}$$

$$\Rightarrow \sin\theta = \frac{\frac{1}{2}\left(p - \frac{1}{p}\right)}{\frac{1}{2}\left(p + \frac{1}{p}\right)} = \frac{p^2 - 1}{p^2 + 1}$$

63. According to question

$$\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

$$\Rightarrow \frac{(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta) - (\cos\theta + \sin\theta)} = \frac{(1 - \sqrt{3}) + (1 + \sqrt{3})}{(1 - \sqrt{3}) - (1 + \sqrt{3})} \quad [\text{Applying componendo and dividendo}]$$

$$\Rightarrow \frac{2\cos\theta}{-2\sin\theta} = \frac{2}{-2\sqrt{3}}$$

$$\Rightarrow \cot\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \tan\theta = \tan 60^\circ \Rightarrow \theta = 60^\circ$$

64. Given,

$$(\cot\theta + \tan\theta) = m \text{ and } (\sec\theta - \cos\theta) = n$$

$$\Rightarrow \left(\frac{1}{\tan\theta} + \tan\theta\right) = m \text{ and } \left(\frac{1}{\cos\theta} - \cos\theta\right) = n$$

$$\Rightarrow \left(\frac{1 + \tan^2\theta}{\tan\theta}\right) = m \text{ and } \frac{(1 - \cos^2\theta)}{\cos\theta} = n$$

$$\Rightarrow \left(\frac{\sec^2\theta}{\tan\theta}\right) = m \text{ and } \frac{(1 - \cos^2\theta)}{\cos\theta} = n$$

$$\Rightarrow m = \frac{1}{\cos^2\theta \times \frac{\sin\theta}{\cos\theta}} \text{ and } \frac{\sin^2\theta}{\cos\theta} = n$$

$$\Rightarrow m = \frac{1}{\cos\theta \sin\theta} \text{ and } n = \frac{\sin^2\theta}{\cos\theta} \dots\dots(1)$$

Now, L.H.S.

$$= (m^2 n)^{\frac{2}{3}} - (mn^2)^{\frac{2}{3}}$$

$$= \left[\frac{1}{\cos^2\theta \sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right]^{\frac{2}{3}} - \left[\frac{1}{\cos\theta \sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right]^{\frac{2}{3}} \quad [\text{from (1)}]$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{\frac{2}{3}} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{\frac{2}{3}} = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta - \tan^2\theta$$

$$= 1 \quad [\because \sec^2\theta - \tan^2\theta = 1]$$

= R.H.S. Hence, Proved.

65. We have,

$$\sin\theta + \sin^2\theta + \sin^3\theta = 1$$

$$\Rightarrow \sin\theta + \sin^3\theta = 1 - \sin^2\theta$$

$$\Rightarrow \sin\theta (1 + \sin^2\theta) = \cos^2\theta$$

$$\Rightarrow \sin^2\theta (1 + \sin^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow (1 - \cos^2\theta) \{1 + (1 - \cos^2\theta)\}^2 = \cos^4\theta$$

$$\Rightarrow (1 - \cos^2\theta) (2 - \cos^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow (1 - \cos^2\theta) (4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$$

$$\Rightarrow 4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^6\theta = \cos^4\theta$$

$$\Rightarrow -\cos^6\theta + 4\cos^4\theta - 8\cos^2\theta + 4 = 0$$

$$\Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$$

66. We have,

$$\text{LHS} = \frac{\left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)}{\left(\frac{1}{\cos^3 A} - \frac{1}{\sin^3 A}\right)}$$

$$\begin{aligned} \Rightarrow \text{LHS} &= \frac{\left(1 + \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A)}{\left(\frac{\sin^3 A - \cos^3 A}{\sin^3 A \cos^3 A}\right)} \\ \Rightarrow \text{LHS} &= \frac{\left(1 + \frac{1}{\sin A \cos A}\right)(\sin A - \cos A) \sin^3 A \cos^3 A}{(\sin^3 A - \cos^3 A)} \\ \Rightarrow \text{LHS} &= \frac{(\sin A \cos A + 1)(\sin A - \cos A) \sin^2 A \cos^2 A}{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)} \quad [\because a^3 - b^3 = (a - b)(a^2 + b^2 + ab)] \\ \Rightarrow \text{LHS} &= \frac{(\sin A \cos A + 1) \sin^2 A \cos^2 A}{(1 + \sin A \cos A)} = \sin^2 A \cos^2 A = \text{RHS} \end{aligned}$$

67. Given, $\sin \theta + 2 \cos \theta = 1$

We have,

$$\begin{aligned} &(\sin \theta + 2 \cos \theta)^2 + (2 \sin \theta - \cos \theta)^2 \\ &= (\sin^2 \theta + 4 \cos^2 \theta + 4 \sin \theta \cos \theta) + (4 \sin^2 \theta + \cos^2 \theta - 4 \sin \theta \cos \theta) \\ &= \sin^2 \theta + 4 \cos^2 \theta + 4 \sin \theta \cos \theta + 4 \sin^2 \theta + \cos^2 \theta - 4 \sin \theta \cos \theta \\ &= 5 \sin^2 \theta + 5 \cos^2 \theta \\ &\Rightarrow 5 (\sin^2 \theta + \cos^2 \theta) \\ &= 5 \\ &\Rightarrow 1^2 + (2 \sin \theta - \cos \theta)^2 = 5 \\ &\Rightarrow (2 \sin \theta - \cos \theta)^2 = 4 \\ &\Rightarrow 2 \sin \theta - \cos \theta = \pm 2 \\ &\Rightarrow 2 \sin \theta - \cos \theta = 2 \end{aligned}$$

68. Given,

$$\left(\frac{x}{a} \sin \theta - \frac{y}{b} \cos \theta\right) = 1 \dots (1)$$

$$\text{and, } \left(\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta\right) = 1 \dots (2)$$

Now, equation (1) is

$$\left(\frac{x}{a} \sin \theta - \frac{y}{b} \cos \theta\right) = 1$$

(Squaring both sides, we get)

$$\frac{x^2}{a^2} \sin^2 \theta + \frac{y^2}{b^2} \cos^2 \theta - \frac{2xy}{ab} \sin \theta \cos \theta = 1 \dots (3)$$

Equation (2) is

$$\left(\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta\right) = 1$$

(Squaring both sides, we get)

$$\frac{x^2}{a^2} \cos^2 \theta + \frac{y^2}{b^2} \sin^2 \theta + \frac{2xy}{ab} \sin \theta \cos \theta = 1 \dots (4)$$

Adding (3) and (4), we get :-

$$\frac{x^2}{a^2} (\sin^2 \theta + \cos^2 \theta) + \frac{y^2}{b^2} (\sin^2 \theta + \cos^2 \theta) = 2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

Hence, Proved.

69. We have,

$$x \sin^3 \theta + y \cos^3 \theta = \sin \theta \cos \theta$$

$$\Rightarrow (x \sin \theta) \sin^2 \theta + (y \cos \theta) \cos^2 \theta = \sin \theta \cos \theta$$

$$\Rightarrow x \sin \theta (\sin^2 \theta) + (x \sin \theta) \cos^2 \theta = \sin \theta \cos \theta \quad [\because x \sin \theta = y \cos \theta]$$

$$\Rightarrow x \sin \theta (\sin^2 \theta + \cos^2 \theta) = \sin \theta \cos \theta$$

$$\Rightarrow x \sin \theta = \sin \theta \cos \theta$$

$$\Rightarrow x = \cos \theta$$

Now, $x \sin \theta = y \cos \theta$

$$\Rightarrow \cos \theta \sin \theta = y \cos \theta \quad [\because x = \cos \theta]$$

$$\Rightarrow y = \sin \theta$$

$$\text{Hence, } x^2 + y^2 = \cos^2 \theta + \sin^2 \theta = 1$$

70. L.H.S.

$$\begin{aligned} &= \frac{\tan A}{1 + \sec A} - \frac{\tan A}{1 - \sec A} \\ &= \tan A \left(\frac{1}{1 + \sec A} - \frac{1}{1 - \sec A} \right) \end{aligned}$$

$$\begin{aligned}
&= \tan A \left(\frac{1 - \sec A - 1 - \sec A}{1 - \sec^2 A} \right) \text{ [Taking L.C.M.]} \\
&= \tan A \left(\frac{-2 \sec A}{-\tan^2 A} \right) \text{ [Since, } \sec^2 A - \tan^2 A = 1 \text{]} \\
&= \frac{2 \sec A}{\tan A} \\
&= \frac{2 \times 1}{\cos A} \times \frac{\cos A}{\sin A} \\
&= \frac{2}{\sin A} \\
&= 2 \operatorname{cosec} A \\
&= \text{R.H.S. Proved.}
\end{aligned}$$

71. We have,

$$\begin{aligned}
\text{LHS} &= \sin^8 \theta - \cos^8 \theta = (\sin^4 \theta)^2 - (\cos^4 \theta)^2 = (\sin^4 \theta - \cos^4 \theta) (\sin^4 \theta + \cos^4 \theta) \\
&\Rightarrow \text{LHS} = (\sin^2 \theta - \cos^2 \theta) (\sin^2 \theta + \cos^2 \theta) (\sin^4 \theta + \cos^4 \theta) \\
&\Rightarrow \text{LHS} = (\sin^2 \theta - \cos^2 \theta) \{ (\sin^2 \theta)^2 + (\cos^2 \theta)^2 + 2 \sin^2 \theta \cos^2 \theta - 2 \sin^2 \theta \cos^2 \theta \} \\
&\Rightarrow \text{LHS} = (\sin^2 \theta - \cos^2 \theta) \{ (\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta \} \\
&\Rightarrow \text{LHS} = (\sin^2 \theta - \cos^2 \theta) (1 - 2 \sin^2 \theta \cos^2 \theta) = \text{RHS}
\end{aligned}$$

72. We have,

$$\begin{aligned}
&\Rightarrow \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A} \\
&\Rightarrow \frac{1}{\operatorname{cosec} A - \cot A} + \frac{1}{\operatorname{cosec} A + \cot A} = \frac{1}{\sin A} + \frac{1}{\sin A} \\
&\Rightarrow \frac{1}{\operatorname{cosec} A - \cot A} + \frac{1}{\operatorname{cosec} A + \cot A} = \frac{2}{\sin A} \\
\text{LHS} &= \frac{1}{\operatorname{cosec} A - \cot A} + \frac{1}{\operatorname{cosec} A + \cot A} \\
&\Rightarrow \frac{\operatorname{cosec} A + \cot A + \operatorname{cosec} A - \cot A}{(\operatorname{cosec} A - \cot A)(\operatorname{cosec} A + \cot A)} \\
&\Rightarrow \frac{2 \operatorname{cosec} A}{\operatorname{cosec}^2 A - \cot^2 A} \\
&\Rightarrow \frac{2}{\frac{\sin A}{1}} = \frac{2}{\sin A} = \text{RHS.}
\end{aligned}$$

Hence Proved.

73. LHS = $\frac{\cos \theta \operatorname{cosec} \theta - \sin \theta \sec \theta}{\cos \theta + \sin \theta}$

$$\begin{aligned}
&= \frac{\cos \theta \times \frac{1}{\sin \theta} - \sin \theta \times \frac{1}{\cos \theta}}{\cos \theta + \sin \theta} = \frac{\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}}{(\cos \theta + \sin \theta)} \\
&= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta (\cos \theta + \sin \theta)} \\
&= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{\sin \theta \cos \theta (\cos \theta + \sin \theta)} \\
&= \frac{\cos \theta}{\sin \theta \cos \theta} - \frac{\sin \theta}{\sin \theta \cos \theta} \\
&= \operatorname{cosec} \theta - \sec \theta = \text{RHS}
\end{aligned}$$

\therefore LHS = RHS

74. We have,

$$\begin{aligned}
&\Rightarrow \text{LHS} = \frac{\cos A}{1 - \sin A} + \frac{\sin A}{1 - \cos A} + 1 \\
&\Rightarrow \text{LHS} = \frac{\cos A(1 - \cos A) + \sin A(1 - \sin A) + (1 - \sin A)(1 - \cos A)}{(1 - \sin A)(1 - \cos A)} \\
&\Rightarrow \text{LHS} = \frac{\cos A - \cos^2 A + \sin A - \sin^2 A + 1 - \sin A - \cos A + \sin A \cos A}{(1 - \sin A)(1 - \cos A)} \\
&\Rightarrow \text{LHS} = \frac{(\cos A + \sin A) - (\cos^2 A + \sin^2 A) + 1 - (\cos A + \sin A) + \sin A \cos A}{(1 - \sin A)(1 - \cos A)} \\
&\Rightarrow \text{LHS} = \frac{(\cos A + \sin A) - 1 + 1 - (\cos A + \sin A) + \sin A \cos A}{(1 - \sin A)(1 - \cos A)} \\
&\Rightarrow \text{LHS} = \frac{\sin A \cos A}{(1 - \sin A)(1 - \cos A)} = \text{RHS}
\end{aligned}$$